Exercise 20

Solve the initial-value problem.

$$3y'' - 2y' - y = 0, \quad y(0) = 0, \quad y'(0) = -4$$

Solution

This is a linear homogeneous ODE with constant coefficients, so its solutions are of the form $y = e^{rx}$.

$$y = e^{rx} \quad \rightarrow \quad \frac{dy}{dx} = re^{rx} \quad \rightarrow \quad \frac{d^2y}{dx^2} = r^2e^{rx}$$

Plug these formulas into the ODE.

$$3(r^2e^{rx}) - 2(re^{rx}) - e^{rx} = 0$$

Divide both sides by e^{rx}.

$$3r^2 - 2r - 1 = 0$$

Solve for r.

$$(3r + 1)(r - 1) = 0$$

$$r = \left\{-\frac{1}{3}, 1\right\}$$

Two solutions to the ODE are $e^{-x/3}$ and e^x. By the principle of superposition, then,

$$y(x) = C_1e^{-x/3} + C_2e^x.$$

Differentiate the general solution.

$$y'(x) = -\frac{1}{3}C_1e^{-x/3} + C_2e^x$$

Apply the initial conditions to determine C_1 and C_2.

$$y(0) = C_1 + C_2 = 0$$

$$y'(0) = -\frac{1}{3}C_1 + C_2 = -4$$

Solving this system of equations yields $C_1 = 3$ and $C_2 = -3$. Therefore, the solution to the initial value problem is

$$y(x) = 3e^{-x/3} - 3e^x.$$

Below is a graph of $y(x)$ versus x.